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# Migdal–Kadanoff Renormalization Group Approach to the Spin-1/2 Anisotropic Heisenberg Model

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The spin-1/2 anisotropic Heisenberg model is studied by generalizing the Migdal–Kadanoff renormalization transformations to quantum spin systems. An approximate one-dimensional decimation is employed besides the potential-moving approximation in this generalization. It is shown that these approximations are valid at high temperatures. The results obtained from these approximations suggest that the two-dimensional spin-1/2 $X-Y$ model shows the critical behavior similar to that expected for the classical $X-Y$ and planar models.

**KEY WORDS:** Spin-1/2 anisotropic Heisenberg model; decimations; Migdal–Kadanoff renormalization transformations; two-dimensional spin-1/2 $X-Y$ model.

## 1. INTRODUCTION

In recent years, many attempts[3–19] have been made to generalize the real space renormalization group approach[1,2] to quantum spin systems such as the Heisenberg, $X-Y$, and transverse Ising models. Especially, many authors[3–8] have been interested in the two-dimensional spin-1/2 $X-Y$ model, because analyses based on high-temperature series[20] suggest that this model without spontaneous magnetization at nonzero temperature[21] shows some kind of phase transition. The computer simulation of this model[22] also supports this suggestion. However, up to now, the renormalization studies[3–8] have given no definite result concerning the critical properties of this model.

In the previous paper[19] we have proposed simple real space renormalization transformations for quantum spin systems. Our approach starts with an approximate decimation for one-dimensional systems. Then, on the basis of it, the Migdal–Kadanoff transformations[23,24] are generalized to

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quantum spin systems in higher dimensions. As mentioned above, the results of the previous renormalization studies on the two-dimensional spin-1/2 $X-Y$ model are not conclusive. Thus, it is of great interest to study this model by the simple approach of the Migdal–Kadanoff transformations.

The results of our approach concerning the critical properties of the spin-1/2 anisotropic Heisenberg model in two and three dimensions seem qualitatively reasonable.[19] On the contrary, Barma *et al.*[25] have pointed out that our approach leads to some strange behaviors in some cases. In this paper, we discuss the nature of the approximations to see to what extent our approach is valid. We also examine the comment of Barma *et al.*[25] Taking the validity of our approach into account, we analyze the results about the two-dimensional spin-1/2 $X-Y$ model.

Throughout this paper, we treat the spin-1/2 anisotropic Heisenberg model in one, two, and three dimensions. The Hamiltonian $\mathcal{H}$ of this model including the factor $-\beta \equiv -1/k_B T$, is given by

$$\mathcal{H} = \sum_{\langle ij \rangle} \left[ K_z \sigma_i^z \sigma_j^z + K_{xy}(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) \right] \tag{1}$$

where $T$ is the temperature and $k_B$ Boltzmann's constant. Here, $\sigma_i^x$, $\sigma_i^y$, and $\sigma_i^z$ denote the Pauli spin operators on the $i$th site of the $d$-dimensional hypercubic lattice and $\sum_{\langle ij \rangle}$ denotes the sum over all nearest-neighbor pairs. The parameters $K_z$ and $K_{xy}$ are written as $K_z = \beta J_z$ and $K_{xy} = \beta J_{xy}$, where $J_z$ is an exchange coupling constant among $z$ components of neighboring spins and $J_{xy}$ is that of $x$ and $y$ components.

The outline of the remainder of this paper is as follows. In Section 2, one-dimensional models are studied by using an approximate decimation. The nature of this approximation is investigated by calculating the leading corrections to this approximation. In Section 3, two- and three-dimensional models are studied by generalizing the Migdal–Kadanoff transformations to quantum spin systems. The validity of this generalization is discussed. Our conclusion and discussion are given in Section 4. Appendix A deals with a derivation of the Migdal–Kadanoff transformations.

## 2. APPROXIMATE DECIMATION FOR ONE-DIMENSIONAL SYSTEMS

Below in this section, we treat only one-dimensional systems. Then, the Hamiltonian (1) can be written as

$$\mathcal{H} = \sum_{i=0}^{N-1} \left[ K_z \sigma_i^z \sigma_{i+1}^z + K_{xy}(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) \right] \tag{2}$$

where $i$ denotes the lattice site of the linear chain with $N+1$ sites.

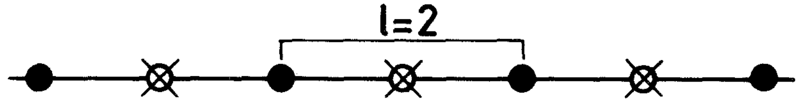



Fig. 1. Decimation in one dimension with scale factor $l = 2$. Circles denote spin operators to be eliminated by the decimation. Dots denote spin operators to remain after the decimation. Lines denote the interactions between neighboring spins. Crosses mean taking a trace over the spin operator marked with a cross.

We consider so-called "decimation" transformation as a renormalization transformation with scale factor $l$:

$$\exp\left[G + \mathcal{H}'(\{\boldsymbol{\sigma}_{lj}\})\right] = \mathrm{Tr}' \exp\left[\mathcal{H}(\{\boldsymbol{\sigma}_i\})\right] \tag{3}$$

where $j = 0, 1, \ldots, N/l$, $i = 0, 1, \ldots, N$ and

$$\mathrm{Tr}' = \prod_{i \neq lj} \mathrm{Tr}_{\sigma_i} \tag{4}$$

This means to take a partial trace over spin operators except every $l$th spin operator (see Fig. 1).

For the Ising model (i.e., $K_{xy} = 0$) in one dimension, this decimation can be carried out exactly. In the quantal case of general anisotropy with $K_{xy} \neq 0$, it is, however, impossible to carry out the decimation exactly even in one dimension because of the noncommutative effect of operators in the Hamiltonian. Thus, we have to make some approximation to obtain an explicit form of the renormalization transformation of interaction parameters from Eq. (3). In the previous paper,[19] we have proposed the following simple approximation to the one-dimensional decimation. Let the Hamiltonian be a sum of nearest-neighbor interactions:

$$\mathcal{H} = \sum_{i=0}^{N-1} H(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{i+1}) \tag{5}$$

where $H(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{i+1})$ denotes a nearest-neighbor interaction between $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_{i+1}$. We divide the Hamiltonian $\mathcal{H}$ into clusters of the nearest-neighbor interactions as

$$\mathcal{H} = \sum_{k=0}^{N/l-1} \left[ \sum_{i=0}^{l-1} H(\boldsymbol{\sigma}_{kl+i}, \boldsymbol{\sigma}_{kl+i+1}) \right] \tag{6}$$

By considering only one cluster, the decimation can be carried out, in principle, as

$$\exp\left[\tilde{G} + H'(\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_l)\right] = \left( \prod_{i=1}^{l-1} \mathrm{Tr}_{\sigma_i} \right) \exp\left[ \sum_{i=0}^{l-1} H(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{i+1}) \right] \tag{7}$$

where $H'(\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_l)$ is a new nearest-neighbor interaction between the remaining spins $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_l$ and $\tilde{G}$ is a constant term independent of $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_l$. Then, using $H'$ and $\tilde{G}$, we construct an approximate transformed Hamiltonian $\mathcal{H}'_A$ and an approximate constant term $G_A$ as follows:

$$\mathcal{H}'_A \equiv \sum_{k=0}^{N/l-1} H'(\boldsymbol{\sigma}_{kl}, \boldsymbol{\sigma}_{(k+1)l}) \tag{8}$$

and

$$G_A \equiv \frac{N}{l} \tilde{G} \tag{9}$$

This process of approximation (see Fig. 2) can be written as

$$\begin{aligned}
\exp(G + \mathcal{H}') &= \mathrm{Tr}' \exp(\mathcal{H}) = \mathrm{Tr}' \exp\left[\sum_{i=0}^{N-1} H(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{i+1})\right] \\
&\simeq \mathrm{Tr}' \prod_{k=0}^{N/l-1} \exp\left[\sum_{i=0}^{l-1} H(\boldsymbol{\sigma}_{kl+i}, \boldsymbol{\sigma}_{kl+i+1})\right] \\
&= \prod_{k=0}^{N/l-1} \exp\left[\tilde{G} + H'(\boldsymbol{\sigma}_{kl}, \boldsymbol{\sigma}_{(k+1)l})\right] \\
&\simeq \exp\left[\frac{N}{l}\tilde{G} + \sum_{k=0}^{N/l-1} H'(\boldsymbol{\sigma}_{kl}, \boldsymbol{\sigma}_{(k+1)l})\right] \\
&= \exp(G_A + \mathcal{H}'_A)
\end{aligned} \tag{10}$$

This approximation takes quantum effect into account within a single cluster. It preserves the form of interactions within nearest-neighbor interactions. It becomes exact, if every $H(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{i+1})$ commutes with each other.

Applying the approximation explained above to the Hamiltonian (2), we obtain the transformed Hamiltonian $\mathcal{H}'_A$ in the same form as Eq. (2) with new parameters $K'_z$, $K'_{xy}$, and $N' = N/l$. For the case of scale factor $l = 2$, we get the following renormalization transformation[19]:

$$K'_z = \tfrac{1}{2}\ln\left[e^{2K_z} + e^{-K_z}C_-(K_z, K_{xy})\right] - \tfrac{1}{4}\ln\left[4e^{-K_z}C_+(K_z, K_{xy})\right] \tag{11}$$

$$K'_{xy} = \tfrac{1}{4}\ln\left[e^{-K_z}C_+(K_z, K_{xy})\right] \tag{12}$$

and

$$\tilde{G} = K'_z + 2K'_{xy} + \ln 2 \tag{13}$$

with

$$C_\pm \equiv \cosh\hat{K} \pm (K_z/\hat{K})\sinh\hat{K} \tag{14}$$

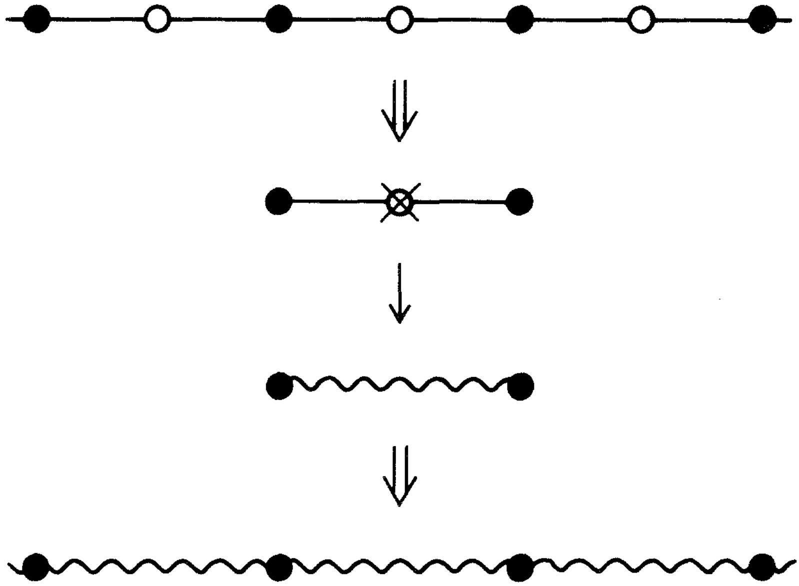



Fig. 2. The approximate one-dimensional decimation. The original system is shown in (a). The decimation is carried out within one cluster shown in (b) to give a new interaction which is denoted by a wavy line in (c). From this new interaction, the renormalized system (d) is constructed.

where

$$\hat{K} \equiv \left(K_z^2 + 8K_{xy}^2\right)^{1/2} \tag{15}$$

We can calculate the free energy $f$ per site of the system from these renormalization transformations by using the well-known formula[2]

$$f(\mathbf{K}) = g(\mathbf{K}) + l^{-d} f(\mathbf{K}') \tag{16}$$

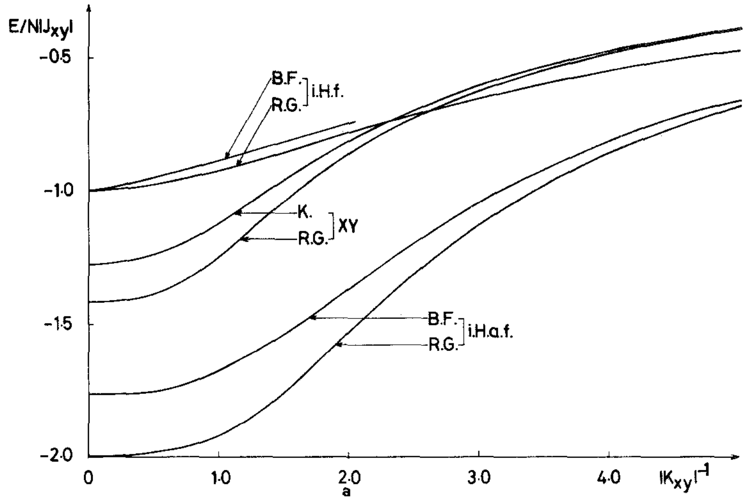
where

$$g(\mathbf{K}) = \frac{1}{N} G(\mathbf{K}) \tag{17}$$

Here, $l = 2$, $d = 1$, and $g(\mathbf{K})$ is given by

$$g(\mathbf{K}) = \frac{1}{l} \tilde{G}(\mathbf{K}) \tag{18}$$

Figures 3a and 3b show the numerical results thus obtained for the internal energy and the specific heat. The exact results obtained for the Ising model



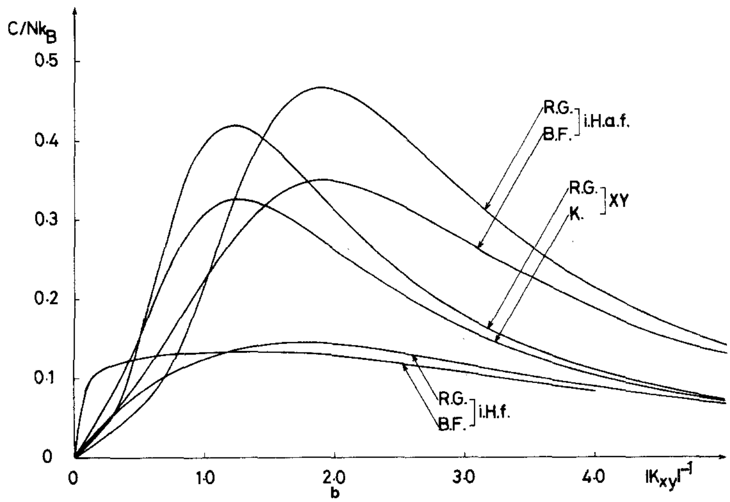




Fig. 3. Temperature dependence of the internal energy (a) and the specific heat (b) of the one-dimensional models. "i.H.f." and "i.H.a.f." denote the isotropic Heisenberg model with ferromagnetic and antiferromagnetic couplings, respectively. "XY" denotes the $X-Y$ model. "R.G." means the present result. "B.F." means the result by Bonner and Fisher.[28] "K." means the exact result by Katsura.[29]

are not shown in these figures. The results shown in Fig. 3b are similar to those obtained by Honda[10] using a different approximation.

In order to investigate the nature of this approximation, we calculate the corrections to the approximation. In Eq. (10), we make two approximations which are represented by $\simeq$. These two approximations are considered as retaining the first-order terms of the following formulas:

$$e^{\lambda(A_1+\cdots+A_n)} = e^{\lambda A_1}\cdots e^{\lambda A_n}e^{B} \tag{19}$$

and

$$e^{\lambda A_1}\cdots e^{\lambda A_n}e^{C} = e^{\lambda(A_1+\cdots+A_n)+D} \tag{20}$$

where

$$B = \sum_{n=2}^{\infty}\lambda^n B_n \tag{21}$$

$$C = \sum_{n=2}^{\infty}\lambda^n C_n \tag{22}$$

$$D = \sum_{n=2}^{\infty}\lambda^n D_n \tag{23}$$

The terms $B$ and $D$ can be derived order by order.[26,27] For example, we have

$$B_2 = -\frac{1}{2}\sum_{i=1}^{n-1}A_i^{\times}(A_{i+1}+\cdots+A_n) \tag{24}$$

$$D_2 = C_2 + \frac{1}{2}\sum_{i=1}^{n-1}A_i^{\times}(A_{i+1}+\cdots+A_n) \tag{25}$$

where we have used Kubo's notation

$$A^{\times}B \equiv AB - BA \tag{26}$$

Using Eqs. (19) and (20), we can calculate the correction terms. First, we calculate $B$ by substituting

$$A_{i+1} = \sum_{j=0}^{l-1} H(\boldsymbol{\sigma}_{il+j}, \boldsymbol{\sigma}_{il+j+1})$$

$\lambda = 1$ and $n = N/l$ into Eq. (19). Here, $A_i$ is explicitly given by

$$\begin{aligned} A_i = {} & K_z(\sigma_{2i-2}^z\sigma_{2i-1}^z + \sigma_{2i-1}^z\sigma_{2i}^z) \\ & + K_{xy}(\sigma_{2i-2}^x\sigma_{2i-1}^x + \sigma_{2i-1}^x\sigma_{2i}^x + \sigma_{2i-2}^y\sigma_{2i-1}^y + \sigma_{2i-1}^y\sigma_{2i}^y) \end{aligned} \tag{27}$$

in the case of $l = 2$. As $A_i^{\times}A_j = 0$ for $i \neq j \pm 1$, the expansion coefficients of

$B = \sum_{n=2}^{\infty} B_n$ with $\lambda = 1$ are given by the following formulas:

$$B_2 = -\tfrac{1}{2} \sum_i A_i^{\times} A_{i+1} \tag{28}$$

$$B_3 = \tfrac{1}{6} \sum_i A_i^{\times 2} A_{i+1} + \tfrac{1}{3} \sum_i A_{i+1}^{\times} A_i^{\times} A_{i+1} \tag{29}$$

$$B_4 = -\sum_i \left( \tfrac{1}{24} A_i^{\times 3} A_{i+1} + \tfrac{1}{8} A_{i+1}^{\times 2} A_i^{\times} A_{i+1} + \tfrac{1}{8} A_{i+1}^{\times} A_i^{\times 2} A_{i+1} + \tfrac{1}{4} A_{i+2}^{\times} A_{i+1}^{\times} A_i^{\times} A_{i+1} \right) \tag{30}$$

where the subscript $n$ of $B_n$ represents the order in $K_z$ and $K_{xy}$. Next, we take a partial trace as

$$\mathrm{Tr}' e^{A_1} \cdots e^{A_n} e^{B} = e^{A'_1} \cdots e^{A'_n} e^{B'} \tag{31}$$

where

$$\begin{aligned} A'_i &= \tilde{G} + H'(\boldsymbol{\sigma}_{(i-1)l}, \boldsymbol{\sigma}_{il}) \\ &= \tilde{G} + K'_z \sigma^z_{2(i-1)} \sigma^z_{2i} + K'_{xy}\left(\sigma^x_{2(i-1)} \sigma^x_{2i} + \sigma^y_{2(i-1)} \sigma^y_{2i}\right) \end{aligned} \tag{32}$$

In Eq. (31), we carry out exactly the calculation within a cluster such as $\mathrm{Tr}_{\boldsymbol{\sigma}_{2i-1}} \mathrm{Tr}_{\boldsymbol{\sigma}_{2i+1}} e^{A_i} e^{A_{i+1}} \sigma^x_{2i-1} \sigma^y_{2i+1}$. Otherwise, we use an expansion in $K'_z$ and $K'_{xy}$ (which are of the second order in $K_z$ and $K_{xy}$), for example in the calculation of terms containing $\exp(-A_i'^{\times})$. The leading order of $B'$ becomes of the second order in $K'_z$ and $K'_{xy}$ (and of the fourth order in $K_z$ and $K_{xy}$). We write this leading-order term as $B'_2$, where the subscript represents the order in $K'_z$ and $K'_{xy}$. Finally, we get correction terms $D$ for $G_A + \mathcal{H}'_A = \sum_{i=1}^{N/l} A'_i$ from $e^{A'_1} \cdots e^{A'_n} e^{B'} = e^{A'_1 + \cdots + A'_n + D}$. By using Eqs. (20) and (25), the leading-order term of $D$ is given by

$$D_2 = B'_2 + \tfrac{1}{2} \sum_i A_i'^{\times} A'_{i+1} \tag{33}$$

We drop the non-Hermitian terms appearing in $D_2$ which should be canceled out with higher-order terms, because the exact $\tilde{\mathcal{H}}'$ is Hermitian in principle. The renormalized parameters $\overline{K}'_z$, $\overline{K}'_{xy}$, and $\tilde{\overline{G}}$ with the leading-order corrections thus obtained are as follows:

$$\begin{aligned} \tilde{\overline{G}} = \tilde{G} &+ 8e^{-2\tilde{G}}\left(K_{xy}^2 C_{xy}^2 + 2K_z K_{xy} C_z C_{xy}\right) \\ &- \tfrac{8}{3} e^{-\tilde{G}}\left[K_{xy}\left(K_z^2 + K_{xy}^2\right) C_{xy} + K_z K_{xy}^2 C_z\right] \\ &- \left(K_{xy}^4 + 2K_z^2 K_{xy}^2\right) \end{aligned} \tag{34}$$

$$\overline{K}'_z = K'_z - \tfrac{8}{3} e^{-\tilde{G}} K_z K_{xy}^2 C_z \tag{35}$$

$$\overline{K}'_{xy} = K'_{xy} - \tfrac{4}{3} e^{-\tilde{G}} K_{xy}\left(K_z^2 + K_{xy}^2\right) C_{xy} \tag{36}$$

where

$$C_z = \tfrac{1}{4}\left[e^{2K_z} - e^{-K_z}\left(\cosh\hat{K} - \frac{K_z}{\hat{K}}\sinh\hat{K}\right)\right] \tag{37}$$

$$C_{xy} = \frac{K_{xy}}{\hat{K}}\, e^{-K_z}\sinh\hat{K} \tag{38}$$

and $\tilde{G}$, $K_z'$, $K_{xy}'$, and $\hat{K}$ are given by Eqs. (11)–(15).

These results show that the noncommutative effects neglected in the approximation start at the fourth order in $K_z$ and $K_{xy}$. Thus, the approximation becomes better at higher temperatures. The results shown in Fig. 3 are in good agreement with the previous results obtained by Bonner and Fisher[28] and by Katsura[29] at high temperatures, as expected.

We can easily show that the correction terms to $\tilde{G}$ are independent of the scale factor $l$ at the purely fourth order in $K_z$ and $K_{xy}$ (which means that no higher-order term is included). In the calculation of the free energy $f$ per site, we use Eq. (16) repeatedly:

$$f(\mathbf{K}) = \sum_{n=0}^{\infty} l^{-n} g(\mathbf{K}^{(n)}) \tag{39}$$

where

$$\mathbf{K}^{(n)} = \mathbf{K}^{(n-1)'} \quad \text{and} \quad \mathbf{K}^{(0)} = \mathbf{K} \tag{40}$$

In Eq. (39), $g(\mathbf{K})$ is given by Eq. (18). Therefore, the leading-order correction to $f$ is proportional to $1/l$. In this sense, we can improve the approximation by increasing the scale factor $l$ at least at high temperatures. If we use large $l$, we can expect further improvement, because the approximation becomes exact in the limit $l \to \infty$ in one dimension.

The results for the free energy of the one-dimensional $X$–$Y$ model obtained by using the approximate renormalization transformations with $l=2$ and $l=3$ are shown in Table I in comparison with the exact results.[29] The results obtained by using Eqs. (34)–(36) are also shown. As mentioned above, the leading-order correction to $\tilde{G}$ is independent of $l$ and the leading-order correction to $f$ is proportional to $1/l$. Therefore, the relation

$$2 \times (f_{\mathrm{ex}} - f_{l=2}) \simeq 3 \times (f_{\mathrm{ex}} - f_{l=3})$$

holds in the leading order, where $f_{\mathrm{ex}}$ denotes the exact free energy and $f_{l=2}$ and $f_{l=3}$ denote the approximate free energies with $l=2$ and $l=3$, respectively. This relation can be rewritten as $f_{\mathrm{ex}} \simeq 3f_{l=3} - 2f_{l=2}$. The values of $3f_{l=3} - 2f_{l=2}$ are also shown in Table I. From the results shown in Table I,

**Table I. Approximate Free Energies and the Exact Free Energy for the One-Dimensional Spin-1/2 $X-Y$ Model[a]**

| $K_{xy}$ | $f_{l=2}$ | $f_{l=3}$ | $f_{l=2'}$ | $3f_{l=3}-2f_{l=2}$ | $f_{ex}$ |
|---|---|---|---|---|---|
| 0.1 | 0.703114 | 0.703109 | 0.703096 | 0.703098 | 0.703098 |
| 0.2 | 0.732625 | 0.732542 | 0.732268 | 0.732375 | 0.732374 |
| 0.3 | 0.780567 | 0.780179 | 0.778298 | 0.779403 | 0.779393 |
| 0.4 | 0.845252 | 0.844161 | 0.836512 | 0.841980 | 0.841906 |
| 0.5 | 0.924621 | 0.922323 | 0.899747 | 0.917728 | 0.917409 |
| 0.6 | 1.016454 | 1.012448 | 0.958888 | 1.004434 | 1.003466 |
| 0.7 | 1.118576 | 1.112460 | 1.003975 | 1.100229 | 1.097914 |
| 0.8 | 1.228992 | 1.220533 | 1.025558 | 1.203616 | 1.198941 |
| 0.9 | 1.345978 | 1.335133 | 1.016235 | 1.313443 | 1.305087 |
| 1.0 | 1.468103 | 1.455013 | 0.972632 | 1.428834 | 1.415208 |

[a] The exact free energy obtained by Katsura[29] is denoted by $f_{ex}$. The approximate free energies $f_{l=2}$ and $f_{l=3}$ are calculated by using the approximate renormalization transformations with $l = 2$ and $l = 3$, respectively. The free energy calculated by using Eqs. (34)–(36) is denoted by $f_{l=2'}$.

we can see that $f_{l=3}$ is an approximation to $f_{ex}$ better than $f_{l=2}$ at high temperatures. Although the free energy $f_{l=2'}$ calculated from Eqs. (34)–(36) gives a right direction of correction to $f_{l=2}$ it breaks down at lower temperatures. The quantity $3f_{l=3}-2f_{l=2}$ is an approximation to $f_{ex}$ better than $f_{l=2}$, $f_{l=3}$, and even $f_{l=2'}$ and it seems to be valid at lower temperatures where $f_{l=2'}$ breaks down. Figures 4a and 4b show the results for the internal energy and the specific heat of the one-dimensional $X-Y$ model obtained from $f_{l=2}$, $f_{l=3}$, and $3f_{l=3}-2f_{l=2}$. The exact results are also shown. The results of $3f_{l=3}-2f_{l=2}$ are better than those of $f_{l=3}$ and $f_{l=2}$ at high temperatures. We must note, however, that the specific heat based on $3f_{l=3}-2f_{l=2}$ becomes worse in the temperature region where the result of $f_{l=3}$ becomes worse than those of $f_{l=2}$.

## 3. THE MIGDAL–KADANOFF TRANSFORMATIONS FOR TWO- AND THREE-DIMENSIONAL SYSTEMS

Below in this section, we consider only the ferromagnetic cases (i.e., $J_z, J_{xy} \geqslant 0$) in two and three dimensions.

Migdal[23] proposed simple renormalization transformations for the classical spin systems. Kadanoff[24] rederived and reinterpreted Migdal's renormalization transformations in a variational method. We generalize these Migdal–Kadanoff transformations to two- and three-dimensional

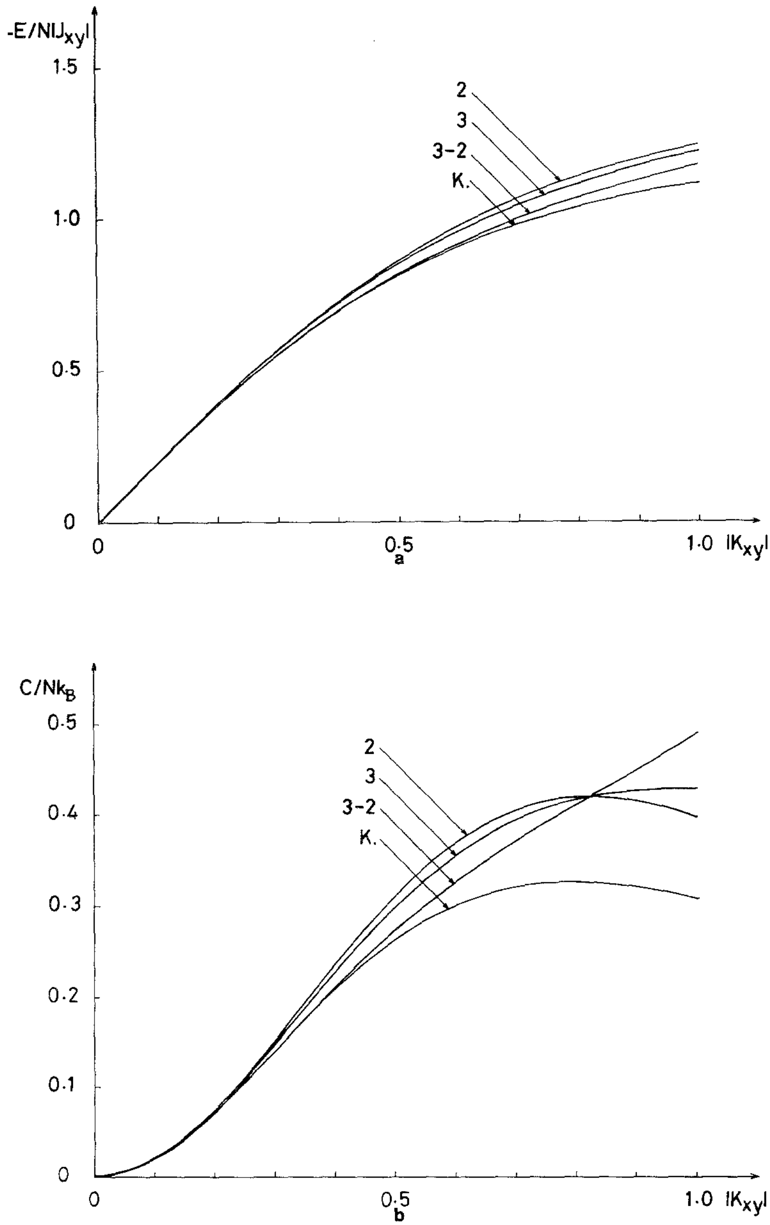



Fig. 4. The internal energy (a) and the specific heat (b) of one-dimensional spin-1/2 $X-Y$ model. "2", "3" and "3-2" denote the results obtained from $f_{l=2}$, $f_{l=3}$, and $3f_{l=3}-2f_{l=2}$, respectively. "K" means the exact result by Katsura.[29]

quantum spin systems, because they are very simple and useful for studying two- and three-dimensional models qualitatively or semiquantitatively.

As is explained in Appendix A, the Migdal–Kadanoff transformations are composed of the two procedures, that is, the potential moving and the one-dimensional decimation. The potential moving can be carried out for quantum spin systems in the same way as for classical systems and it gives a lower bound approximation for the free energy. It is, however, impossible to carry out the one-dimensional decimation exactly for quantum spin systems. Thus, we apply the same approximation as described in Section 2 to this one-dimensional decimation.

We label the direction of the axes of the $d$-dimensional hypercubic lattice as $1, 2, \ldots,$ and $d$; the one-dimensional decimation in each direction is successively performed in order of this direction number (see Appendix A). Then, we obtain, in the case of $l = 2$, the renormalization transformation of the coupling constants $K_{z1}$ and $K_{xy1}$ of 1-direction of the $d$-dimensional hypercubic lattice as

$$K'_{z1} = 2^{d-1} \cdot \tfrac{1}{2} \ln\left[e^{2K_{z1}} + e^{-K_{z1}} C_-(K_{z1}, K_{xy1})\right] - 2^{d-1} \cdot \tfrac{1}{4} \ln\left[4e^{-K_{z1}} C_+(K_{z1}, K_{xy1})\right] \tag{41}$$

and

$$K'_{xy1} = 2^{d-1} \cdot \tfrac{1}{4} \ln\left[e^{-K_{z1}} C_+(K_{z1}, K_{xy1})\right] \tag{42}$$

where $C_\pm$ are defined by Eqs. (14) and (15) (see Appendix A).

First, we summarize the results obtained from this transformation concerning the critical properties of two- and three-dimensional systems and compare them with the results obtained from high-temperature series to see that this approach gives qualitatively reasonable results.

For $d = 2$, this renormalization transformation has the following nontrivial fixed points:

(1) an Ising fixed point at $K^*_{z1} \simeq 0.61$ and $K^*_{xy1} = 0$;
(2) an $X$–$Y$ fixed point at $K^*_{z1} \simeq 0.58$ and $K^*_{xy1} \simeq 1.31$.

For $d = 3$, the above transformations (41) and (42) have the following nontrivial fixed points:

(1) an Ising fixed point at $K^*_{z1} \simeq 0.26$ and $K^*_{xy1} = 0$;
(2) an isotropic Heisenberg fixed point at $K^*_{z1} = K^*_{xy1} \simeq 0.34$;
(3) an $X$–$Y$ fixed point at $K^*_{z1} \simeq 0.02$ and $K^*_{xy1} \simeq 0.28$.

Figure 5a shows the critical lines determined from this renormalization transformation for two dimensions. Figure 5b shows the critical lines for

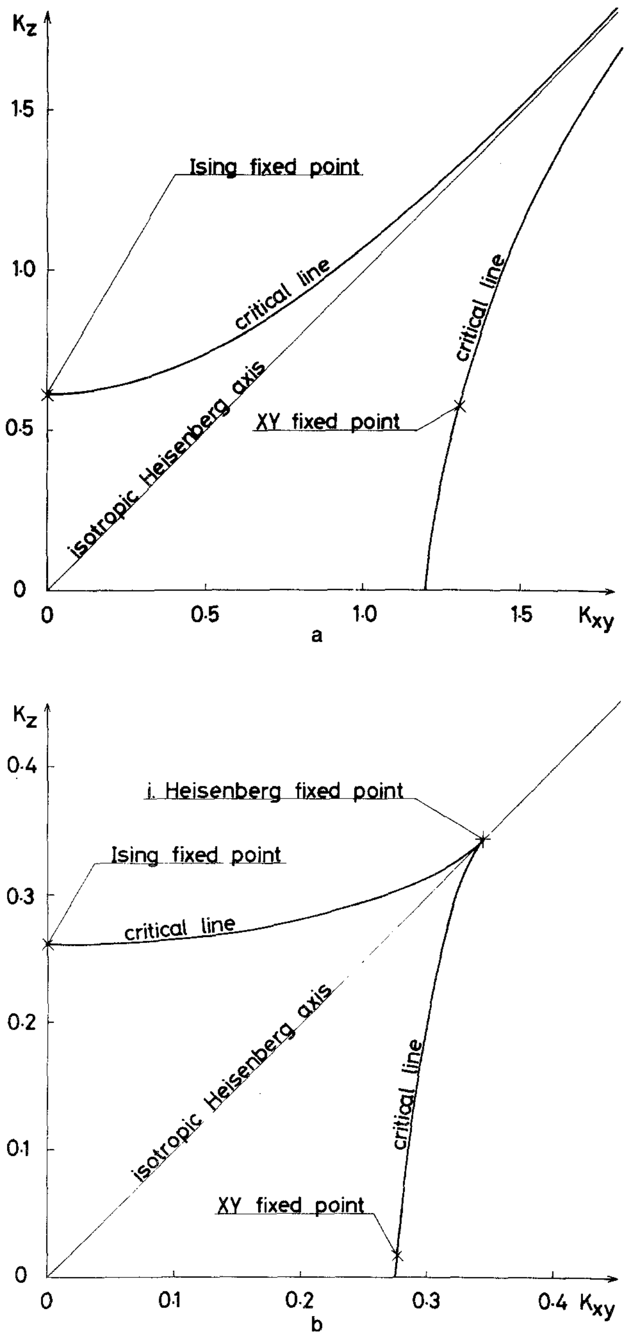



Fig. 5. Critical lines of the anisotropic Heisenberg model in two dimensions (a) and in three dimensions (b).

three dimensions. The critical behavior for $K_z > K_{xy}$ is controlled by the Ising fixed point and for $K_z < K_{xy}$ by the $X-Y$ fixed point in both two and three dimensions. In three dimensions, the critical behavior of the isotropic Heisenberg model (i.e., $K_{xy} = K_z$) is controlled by the isotropic Heisenberg fixed point.[19] Thus, the present model is classified into three universality classes, namely, the Ising, isotropic Heisenberg, and $X-Y$ types, as it should be.

For the isotropic Heisenberg model, there is a nontrivial fixed point in three dimensions. On the contrary, there is no nontrivial fixed point in two dimensions. In other words, the isotropic Heisenberg model undergoes a phase transition at a finite temperature in three dimensions, but it undergoes no phase transition at any nonzero temperature in *two dimensions*. This result agrees well with those obtained from the high-temperature series expansions.[30,31]

The critical lines for two dimensions shown in Fig. 5a and for three dimensions shown in Fig. 5b agree very well with those obtained by the high-temperature series[32] in that the critical temperature is almost constant in the region $0 \leqslant J_{xy}/J_z \lesssim 0.7$ for the square lattice and in the region $0 \leqslant J_{xy}/J_z \lesssim 0.8$ for the simple cubic lattice.

Next, we calculate critical exponents. Linearizing the renormalization transformation (41) and (42) around a nontrivial fixed point $K_{z1} = K_{z1}^*$ and $K_{xy1} = K_{xy1}^*$, we obtain

$$\begin{pmatrix} K'_{z1} \\ K'_{xy1} \end{pmatrix} - \begin{pmatrix} K^*_{z1} \\ K^*_{xy1} \end{pmatrix} = \tilde{T}_1^* \left\{ \begin{pmatrix} K_{z1} \\ K_{xy1} \end{pmatrix} - \begin{pmatrix} K^*_{z1} \\ K^*_{xy1} \end{pmatrix} \right\} \tag{43}$$

with

$$\tilde{T}_1^* = \left[ \begin{matrix} \dfrac{\partial K'_{z1}}{\partial K_{z1}} & \dfrac{\partial K'_{z1}}{\partial K_{xy1}} \\[2ex] \dfrac{\partial K'_{xy1}}{\partial K_{z1}} & \dfrac{\partial K'_{xy1}}{\partial K_{xy1}} \end{matrix} \right]_{\substack{K_{z1} = K^*_{z1} \\ K_{xy1} = K^*_{xy1}}} \tag{44}$$

where $K'_{z1}$ and $K'_{xy1}$ are defined by Eqs. (41) and (42). Then we calculate the eigenvalues of the matrix $\tilde{T}_1^*$ for each fixed point. Only one of two eigenvalues is relevant (i.e., greater than unity) for the Ising and $X-Y$ fixed points in two and three dimensions. The thermal exponent $y_T$ is determined from this relevant eigenvalue $\lambda_T$ by the definition

$$y_T = \ln \lambda_T / \ln l \tag{45}$$

where we have used the scale factor $l = 2$ in this case.

For the isotropic Heisenberg fixed point, two eigenvalues are both relevant in three dimensions. One of these eigenvalues is associated with a scaling field which measures the deviation from the isotropic Heisenberg axis in the parameter space. This eigenvalue $\lambda_g$ does not enter the critical behavior of the isotropic Heisenberg model. However, this eigenvalue $\lambda_g$ determines the crossover behavior from the isotropic Heisenberg model to the Ising model or the $X-Y$ model; it determines how a point being out of the isotropic Heisenberg axis but near the isotropic Heisenberg fixed point leaves this fixed point and approaches one of the other fixed points, as the renormalization transformation goes on. Hence, we define the anisotropy exponent $y_g$ as

$$y_g = \ln \lambda_g / \ln l \tag{46}$$

On the contrary, the other relevant eigenvalue is related to the distance from a point on the isotropic Heisenberg axis to the isotropic Heisenberg fixed point. Therefore, this eigenvalue $\lambda_T$ determines the critical behavior of the isotropic Heisenberg model. From this eigenvalue $\lambda_T$, the thermal exponent $y_T$ is determined by Eq. (45). From these two exponents $y_g$ and $y_T$, the crossover exponent $\phi$[33,34] is determined as

$$\phi = y_g / y_T \tag{47}$$

Table II shows the exponents thus obtained for each fixed point.

In order to calculate the magnetic exponent $y_H$, we consider the Hamiltonian with a magnetic field as

$$\mathcal{H} = \sum_{\langle ij \rangle} \left[ K_z \sigma_i^z \sigma_j^z + K_{xy} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) \right] + h_\alpha \sum_i \sigma_i^\alpha \tag{48}$$

where $h_\alpha$ denotes a magnetic field coupled to the $\alpha$ component of spins and $\alpha$ denotes $x$ or $z$. In order to apply the Migdal–Kadanoff transformations to this Hamiltonian, we have to generalize the one-dimensional decimation and the potential moving to the Hamiltonian with a magnetic field. These

**Table II. Nontrivial Fixed Points and Exponents in Two and Three Dimensions**

| Fixed Point | | Position | | Exponents | | | |
|---|---|---|---|---|---|---|---|
| Dimension | Type | $K_{z1}^*$ | $K_{xy1}^*$ | $y_T$ | $y_g$ | $y_{H_x}$ | $Y_{H_z}$ |
| $2d$ | Ising | 0.61 | 0 | 0.75 | — | $<0$ | 1.81 |
| | $X-Y$ | 0.58 | 1.31 | 0.16 | — | 1.68 | $<0$ |
| $3d$ | Ising | 0.26 | 0 | 0.94 | — | $<0$ | 2.12 |
| | Isotropic Heisenberg | 0.34 | 0.34 | 0.72 | 1.12 | 2.06 | 2.06 |
| | $X-Y$ | 0.02 | 0.28 | 0.86 | — | 2.08 | $<0$ |

generalizations are given in Appendix A. Using these generalizations, we can construct the renormalization transformation including a magnetic field. To the first order of the magnetic field, the renormalization transformations of $K_z$ and $K_{xy}$ are unchanged. At a nontrivial fixed point $K_z = K_z^*$ and $K_{xy} = K_{xy}^*$, the renormalization transformation of the magnetic field $h_\alpha$ takes, to the first order, the form

$$h'_\alpha = \lambda_\alpha h_\alpha \tag{49}$$

where $\lambda_\alpha$ is determined from $K_z^*$ and $K_{xy}^*$ and $\alpha$ denotes $x$ or $z$. From this eigenvalue $\lambda_\alpha$, the magnetic exponent $y_{H_\alpha}$ is given by

$$y_{H_\alpha} = \ln \lambda_\alpha / \ln l \tag{50}$$

For the Ising fixed point in two and three dimensions, we have that $y_{H_x} < 0$ and $y_{H_z} > 0$. That is, a magnetic field transverse to the Ising axis is irrelevant and a magnetic field parallel to the Ising axis is relevant. On the contrary, we have that $y_{H_x} > 0$ and $y_{H_z} < 0$ for the $X-Y$ fixed point in two and three dimensions. That is, a magnetic field parallel to the $X-Y$ plane is relevant and a magnetic field perpendicular to the $X-Y$ plane is irrelevant. For the isotropic Heisenberg fixed point in three dimensions, we have $y_{H_x} = y_{H_z} > 0$. These results are also shown in Table II. The above results for the Ising fixed point in two and three dimensions show that a sufficiently small transverse magnetic field does not change the critical exponents of the Ising model. This agrees with such a result being suggested from the series expansion method[35] that the transverse field changes only the critical temperature without changing critical exponents until the critical field is reached. The above result for the $X-Y$ fixed point shows that a sufficiently small magnetic field perpendicular to the $X-Y$ plane does not change the critical exponents of the $X-Y$ model. This seems to be reasonable from the analogy of the Ising model. It is one of the great merits of our simple renormalization group method that the above physical results are obtained quite easily in our approximation.

The renormalization transformation of the coupling constants of the 1-direction is given by Eqs. (41) and (42). However, the renormalization transformation of the coupling constants of the 2-direction is different from Eqs. (41) and (42). This problem of anisotropy in the Migdal–Kadanoff transformations is explained in Appendix A. If the coupling constants $\mathbf{K}_m = \binom{K_{zm}}{K_{xym}}$ of the $m$-direction satisfy the condition

$$\mathbf{K}_1 = l\mathbf{K}_2 = \cdots = l^{m-1}\mathbf{K}_m = \cdots = l^{d-1}\mathbf{K}_d \tag{51}$$

$l$ being the scale factor, then the renormalized coupling constants $\mathbf{K}'_m$ also satisfy this condition. In this case, the renormalization transformation of each $\mathbf{K}_m$ is essentially reduced to Eqs. (41) and (42) and makes no trouble.

Therefore, we regard the Migdal–Kadanoff transformations as renormalization transformations for a model whose coupling constants satisfy the condition (51). See Appendix A.

In order to compare the results obtained by this approach with those by other theories which usually study the case of isotropic coupling constants (i.e., $\mathbf{K}_1 = \mathbf{K}_2 = \cdots = \mathbf{K}_d$), we consider the following procedure. First, we consider a model with isotropic coupling constants $\tilde{\mathbf{K}}$. Then, from this model, we construct a model whose coupling constants satisfy the condition (51) by the potential moving. According to the variational method,[24] the potential moving has to conserve the total magnitude of coupling constants of any interaction. This means that the new anisotropic coupling constants $\mathbf{K}_m$ have to satisfy the condition

$$\sum_{m=1}^{d} \mathbf{K}_m = d\tilde{\mathbf{K}} \tag{52}$$

From Eqs. (51) and (52), the relation between $\mathbf{K}_m$ and $\tilde{\mathbf{K}}$ is given by

$$\mathbf{K}_1 = l^{m-1}\mathbf{K}_m = \frac{l^{d-1}(l-1)}{l^d - 1} d\tilde{\mathbf{K}} \tag{53}$$

Using this relation, we can compare the results obtained by the Migdal–Kadanoff transformations with those by other theories.

In Table III, we compare the results obtained by our new method concerning the critical points and exponents $y_T$, $y_H$, and $\phi$ for the Ising, isotropic Heisenberg, and $X-Y$ models in two and three dimensions with those obtained by other approaches. For the Ising, isotropic Heisenberg, and $X-Y$ models in three dimensions, the critical inverse temperature $\tilde{K}_c$ or $K_{1c}$ for each model is in rough agreement with that obtained from the high-temperature series.[36,31,37] The exponents $y_T$ and $y_H$ are in poor agreement. The qualitative features are, however, in good agreement: The relative magnitudes of the critical values $\tilde{K}_c$ or $K_{1c}$ for these models agree with the results obtained from the high-temperature series, respectively. According to the analyses by the high-temperature series, the Ising model has the largest thermal exponent $y_T^{(I)}$, the $X-Y$ model has the next largest $y_T^{(XY)}$, and the isotropic Heisenberg model has the smallest $y_T^{(H)}$. That is, the singularity of the specific heat of each model is stronger in this order. Our results about the thermal exponent $y_T$ show a similar tendency, namely,

$$y_T^{(I)} > y_T^{(XY)} > y_T^{(H)}, \quad \text{i.e.,} \quad \alpha^{(I)} > \alpha^{(XY)} > \alpha^{(H)} \tag{54}$$

through $2 - \alpha = d/y_T$. Similarly we obtain $\nu^{(I)} < \nu^{(XY)} < \nu^{(H)}$ through $\nu = y_T^{-1}$. These inequalities concerning critical exponents were derived in a plausible way by one of the authors.[38] The high-temperature series

**Table III. The Critical Couplings and Critical Exponents of the Ising, Isotropic Heisenberg, and $X-Y$ Models in Two and Three Dimensions[a]**

| | | | Renormalization | Series or exact |
|---|---|---|---|---|
| $2d$ | The Ising model | $K_c$ | $K_{1c} \simeq 0.61$<br>$\tilde{K}_c \simeq 0.46$ | 0.44 (exact)[39] |
| | | $y_T$ | 0.75 | 1 (exact)[39] |
| | | $y_H$ | 1.81 | 1.875 (exact)[40] |
| | The $X-Y$ model | $K_c$ | $K_{1c} \simeq 1.20$<br>$\tilde{K}_c \simeq 0.90$ | 0.64 (series)[20] |
| | | $y_T$ | 0.16 | $\gamma \simeq 2.5$ (series)[20] |
| | | $y_H$ | 1.68 | $\Delta \simeq 2.8$ (series)[20] |
| $3d$ | The Ising model | $K_c$ | $K_{1c} \simeq 0.26$<br>$\tilde{K}_c \simeq 0.15$ | 0.22 (series)[36] |
| | | $y_T$ | 0.94 | 1.6 (series)[36] |
| | | $y_H$ | 2.12 | 2.5 (series)[36] |
| | The isotropic Heisenberg model | $K_c$ | $K_{1c} \simeq 0.34$<br>$\tilde{K}_c \simeq 0.20$ | 0.30 (series)[31] |
| | | $y_T$ | 0.72 | 1.4 (series)[31] |
| | | $y_H$ | 2.06 | 2.5 (series)[31] |
| | | $\phi$ | 1.56 | 1.25 (series)[34] |
| | The $X-Y$ model | $K_c$ | $K_{1c} \simeq 0.28$<br>$\tilde{K}_c \simeq 0.16$ | 0.25 (series)[37] |
| | | $y_T$ | 0.86 | 1.5 (series)[37] |
| | | $y_H$ | 2.08 | 2.5 (series)[37] |

[a] For three-dimensional models, hyperscaling relations are assumed in expressing the exponents obtained from series analyses[31,36,37] in terms of $y_T$ and $y_H$. For the two-dimensional $X-Y$ model, the exponents $\gamma$ and $\Delta$ estimated from series expansions[20] are shown here instead of $y_T$ and $y_H$, because the series analysis[20] suggests that hyperscaling may be violated in this model.

expansions suggest that the magnetic exponent $y_H$, consequently the exponent $\delta$, takes almost the same value for the Ising, isotropic Heisenberg, and $X-Y$ models in three dimensions. The values of the present magnetic exponents $y_H$, for these models are also very close to each other.

For the Ising model in two dimensions, the critical value $\tilde{K}_c \simeq 0.46$ is in very good agreement with the exact value $K_c \simeq 0.44$.[39] Though the thermal exponent $y_T \simeq 0.75$[24] is not so good in comparison with the exact value $y_T = 1.0$,[39] the magnetic exponent $y_H \simeq 1.81$ is in good agreement with the exact value $y_H = 1.875$.[40]

As mentioned above, the present simple renormalization group approach gives qualitatively reasonable results for two- and three-dimensional models except for the two-dimensional $X-Y$ model.

For the $X-Y$ model in two dimensions, the critical value $\tilde{K}_c \simeq 0.9$ seems to be too large compared with the value $K_c \simeq 0.64$ estimated from the high-temperature series.[20] The thermal exponent $y_T$ obtained from our approach is fairly small; this means that the singularity of the specific heat is very weak. This agrees with the proposition that the specific heat is nonsingular, which has been confirmed from the high-temperature series.[20] The fact that $y_T$ is very small also means that the corresponding eigenvalue $\lambda_T$ is semimarginal, that is, $\lambda_T \sim 1$. The existence of a marginal scaling field suggests the existence of the fixed line; this fixed line is expected to appear for the classical $X-Y$ and planar models.[41–49] Therefore, this semimarginal thermal exponent $y_T$ suggests that the two-dimensional spin-1/2 $X-Y$ model shows the critical behavior of the same type as the classical models do. This small value of $y_T$ also leads to a large value of the critical exponent $\gamma$ of the susceptibility; this suggests an exponential divergence of the susceptibility. Though the susceptibility is expected to diverge exponentially for the classical models,[43–47] a usual power law with the exponent $\gamma \simeq 2.5$ is suggested from the high-temperature series for the spin-1/2 $X-Y$ model.[20] This small value of $y_T$ is not consistent with the above prediction obtained from the high-temperature series about the susceptibility.

Next, we argue the validity of our approach. As explained before, the transformation is obtained by using two approximations, that is, the potential moving approximation and the cluster approximation used in Section 2 for the one-dimensional decimation. The potential moving approximation for classical systems becomes exact in the two limits of infinite and zero temperatures.[24] However, generally speaking, the potential moving becomes exact only in the limit of infinite temperature in the quantal case of general anisotropy with $J_{xy} \neq 0$. In the special case $J_z > J_{xy}$, we can prove that the free energy obtained by using the potential moving agrees with the exact one in the zero-temperature limit. After all, the potential moving approximation for quantum systems is a high-temperature approximation and the corrections to it start at the second order in $K_z$ and $K_{xy}$ in general.[24,50] The cluster approximation used in Section 2 for one-dimensional decimation is also an approximation good at high temperatures. By calculating the correction terms in the same way as in Section 2, we can show that the corrections to the cluster approximation for the

Migdal–Kadanoff transformation in two dimensions start at the fourth order in $K_z$ and $K_{xy}$. Thus, both approximations used in the present paper are high-temperature approximations. Thus, the present approximate method is an approach good at high temperatures, just as other renormalization group approaches to quantum systems are.[3–10]

Barma *et al.*[25] have pointed out in their comment that the present approach leads to strange behaviors in some cases. We examine the comment of Barma *et al.* Their comment can be summarized as follows:

1. There is no $X-Y$ fixed point at zero temperature in any dimension.
2. The nontrivial $X-Y$ fixed point $X^*$ continues to exist even for $d<2$.
3. A new $X-Y$ fixed point $Y^*$ which is stable emerges from zero temperature as $d$ decreases below 2.
4. The zero-temperature entropy of the isotropic Heisenberg model given by the transformation is incorrect.
5. The lower critical dimensionality $d_c$ for the isotropic Heisenberg model depends on the scale factor $l$ ($d_c=2$ which is exact for $l=2$, $d_c \simeq 2.11$ for $l=3$, and $d_c$ becomes the worse for the larger $l$).

As mentioned before, the present approach is a high-temperature approximation. The above points 1, 4, and 5 concern the nature of the system at zero temperature, where the present approach is invalid. Therefore, the present approach may give incorrect results at zero temperature, which does *not* mean that the present approach is invalid at higher temperatures. We must note further about comment 5 that, for classical systems, the Migdal–Kadanoff transformations become exact in the two limits of zero and infinite temperatures. Therefore, the Migdal–Kadanoff transformations give the correct lower critical dimensionality $d_c$ independent of the scale factor $l$ for classical systems. For quantum systems, however, the present approach becomes exact only in the limit of infinite temperature. Therefore, it cannot give the correct $d_c$. We should be satisfied with the result that there is no fixed point for the two-dimensional isotropic Heisenberg model in the high-temperature region where the present approach is valid. Furthermore, the present approach for dimensions higher than one is not improved by increasing the scale factor $l$, though the present approach for one dimension is improved for larger $l$, as shown in Section 2. The reasons are as follows. First, the amount of the potential moved is increased by increasing the scale factor $l$, which is considered to make the approximation worse. Next, there are always the bonds not included in the clusters in which the approximate decimation is performed, while each bond belongs to one of the clusters in one dimension. Therefore, the amount of the quantum effects neglected is not reduced by increasing the scale factor $l$ for dimensions higher than one. Thus, it has less meaning to make $l$ larger.

The points 2 and 3 are crucial. The unstable $X-Y$ fixed point $X^*$ and the stable $X-Y$ fixed point $Y^*$ in dimensions lower than two may be superficial. Therefore, the fixed point $X^*$ in two dimensions may be also unreliable, in the sense that the fixed point $X^*$ is located at too low temperature for the high-temperature approximation to be reliable. This does *not* mean that the approximation is invalid in the whole range of temperatures. As mentioned before, the approximation is valid at least at high temperatures. A similar situation occurs for some other renormalization group approaches to quantum spin systems.[3,6,8,9] That is, these approaches also yield a superficial fixed point (an isotropic Heisenberg fixed point in two dimensions) in the temperature region where these approaches are considered to be invalid. Another possible interpretation of a pair of fixed points such as $X^*$ and $Y^*$ by Tatsumi[51] is discussed in the next section.

Though the $X-Y$ fixed point $X^*$ in two dimensions may be unreliable, the marginal critical behavior suggested from this fixed point is *not* totally unreliable: This semimarginal critical behavior can be seen from the flow of the renormalization transformation at higher temperatures, where the present approach is expected to be valid. Figures 6a and 6b show the velocity of the flow of the renormalization transformation in two and three dimensions, respectively. By the semimarginal critical behavior we mean that the velocity in these figures approaches the horizontal axis which represents the zero velocity in the manner that it is nearly parallel to the horizontal axis. Such an aspect can be seen from Fig. 6a for the two-dimensional $X-Y$ model. The point is that this aspect is distinct even at higher temperatures (i.e., $|\mathbf{K}_1| \sim 0.6$–$0.8$). Comparing the velocity flow line of the two-dimensional $X-Y$ model with those of the Ising and Heisenberg models in two dimensions, as shown in Fig. 6a, and comparing Fig. 6a for two dimensions with Fig. 6b for three dimensions, we can clearly see the marginal nature of the two-dimensional $X-Y$ model even from the high-temperature part of the figures. Thus, the results of the present approach suggest that the two-dimensional spin-$1/2$ $X-Y$ model shows the marginal critical behavior which is expected for the two-dimensional classical $X-Y$ and planar models, though the critical inverse temperature $K_c$ and the exponents $y_T$ and $y_H$ obtained from the $X-Y$ fixed point $X^*$ in two dimensions may be unreliable.

The qualitatively reasonable results concerning the critical properties of the two-dimensional Ising model and of the Ising, $X-Y$, and isotropic Heisenberg models in three dimensions are obtained from the fixed points which are located at high temperatures where the present approach is expected to be valid.

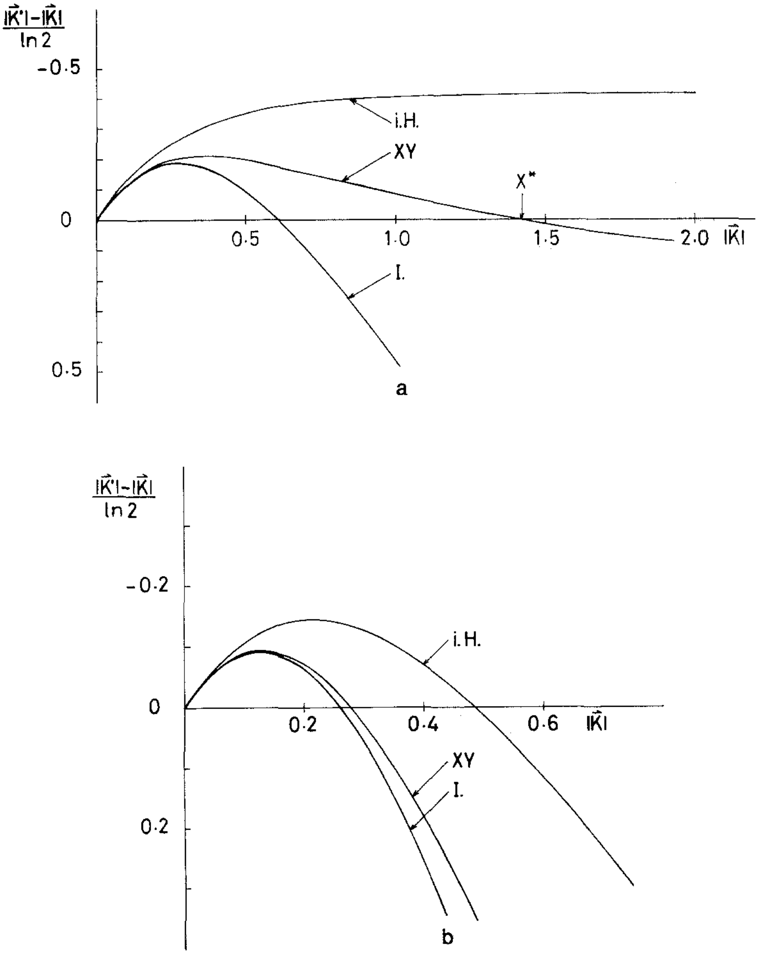



Fig. 6. Velocity of the flow of the renormalization transformation, which is measured by $(|\mathbf{K}'| - |\mathbf{K}|)/\ln 2$, where $|\mathbf{K}| = (K_{z1}^2 + K_{xy1}^2)^{1/2}$. Figures (a) and (b) show the results for two and three dimensions, respectively. "i.H.", "XY" and "I." denote the isotropic Heisenberg, $X-Y$ and Ising models, respectively. For the $X-Y$ model, we calculate $(|\mathbf{K}'| - |\mathbf{K}|)/\ln 2$ and $|\mathbf{K}|$ after applying the renormalization transformation to the pure $X-Y$ model (i.e., $K_z = 0$) a few times.

## 4. CONCLUSION AND DISCUSSION

We have proposed a simple approximate decimation method for one-dimensional quantum spin systems. The spin-1/2 anisotropic Heisenberg model is studied in this approximation. We also calculate the corrections to this approximation to show that the approximation is a high-temperature approximation and that it is improved by increasing the scale factor $l$ at least at high temperatures. The thermodynamic properties obtained from the present approximation are in good agreement with those obtained by Bonner and Fisher[28] and by Katsura[29] at high temperatures.

Using this approximation, we have generalized the Migdal–Kadanoff transformations to quantum spin systems. By the help of this generalized Migdal–Kadanoff transformation with the scale factor $l = 2$, the two- and three-dimensional spin-1/2 anisotropic Heisenberg models are studied. This simple renormalization approach gives qualitatively reasonable results for two- and three-dimensional models except for the two-dimensional $X-Y$ model. This approach for higher dimensions is a high-temperature approach in general.

The unstable $X-Y$ fixed point $X^*$ for two dimensions continues to exist in dimensions lower than two, and a new stable $X-Y$ fixed point $Y^*$ emerges from zero temperature as the dimension decreases below two, as pointed out by Barma *et al.*[25] These fixed points in dimensions lower than two may be superficial. Therefore, the fixed point $X^*$ in two dimensions may be unreliable in the sense that it lies at low temperature where the present approach may be invalid. However, from the renormalization flow in the high-temperature region where the present approach is considered to be valid, we can see the marginal nature of the critical behavior of the two-dimensional spin-1/2 $X-Y$ model. Thus, it is suggested that the two-dimensional spin-1/2 $X-Y$ model shows the critical behavior similar to that expected for the classical $X-Y$ and planar models.

Recently, Tatsumi[51] proposed a modification of the present approach to construct a transformation which transforms the pure $X-Y$ model (i.e., $K_z = 0$) to itself. He also obtained a pair of stable and unstable fixed points. He interpreted these fixed points as follows: As the dimensionality $d$ decreases, these unstable and stable fixed points approach each other and merge into a marginal fixed point at the critical dimensionality $d_c$. He obtained the result that $d_c$ thus determined is close to two. If we adopt this interpretation, the result for the present approach is as follows: The fixed points $X^*$ and $Y^*$ merge into a marginal fixed point at the dimensionality $d_c \simeq 1.95$, and the critical inverse temperature $K_{xy1_c}$ for the pure $X-Y$ model is approximately 2.5 in this case. See Fig. 7. The fact that $d_c$ thus determined is very close to two seems to support our interpretation concerning the two-dimensional spin-1/2 $X-Y$ model mentioned above, though the value of $K_{xy1_c}$ is too large to be reliable in the present approach.

There exists a similar situation in the approach of Dekeyser *et al.*[8]: They determined the value $p_c$ of the parameter $p$ of the linear renormalization transformation in two dimensions so that the $X-Y$ fixed point may be a marginal one, and obtained the conclusion with which our interpretation

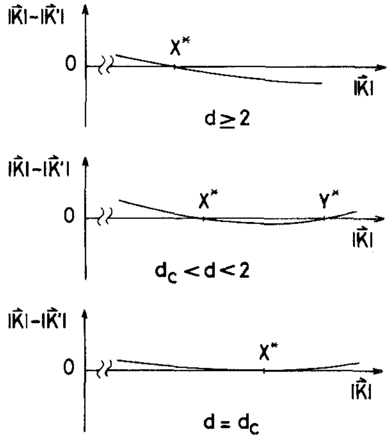



Fig. 7. The fixed points $X^*$ and $Y^*$ approach each other as the dimensionality $d$ decreases. They merge into a marginal fixed point at $d = d_c$.

is consistent. Under the condition $p < p_c$, their approach gives a pair of unstable and stable fixed points (see Fig. 1 of Ref. 8) as in the present case for $d_c < d < 2$. Thus, Dekeyser *et al.* adjusted the parameter $p$ in order to get a marginal $X-Y$ fixed point, while Tatsumi adjusted the dimensionality $d$.

As an application of our approach to other quantum systems, we have calculated the thermodynamic properties of the half-filled Hubbard model in one dimension by using the approximate decimation explained in Section 2. Details of this calculation will be reported in the near future.[52]

## APPENDIX A: THE MIGDAL–KADANOFF TRANSFORMATIONS

In this appendix, we give a derivation of the Migdal–Kadanoff transformations, following Kadanoff.[24]

Kadanoff derived the potential moving as an approximation which gives a lower bound of the exact free energy by using a variational method.[24] This variational method is valid also for the quantum spin systems, because the Hamiltonian of the system is Hermitian. According to the variational method, we can move potential terms from one set of bonds in the lattice to equivalent bonds, but we must not change the total amount of any type of potential. This is called "potential moving."

We consider the following Hamiltonian on the $d$-dimensional hypercubic lattice:

$$\mathcal{H} = \sum_{m=1}^{d} \sum_{\langle ij \rangle \in m} \{ K_{zm}\sigma_i^z\sigma_j^z + K_{xm}\sigma_i^x\sigma_j^x + K_{ym}\sigma_i^y\sigma_j^y \} + h\sum_i \sigma_i^z \quad \text{(A.1)}$$

where $\sigma_i^z$, $\sigma_i^x$, and $\sigma_i^y$ denote the Pauli spin operators on the $i$th site, $m = 1, 2, \ldots, d$ label the directions of the nearest-neighbor bonds and $\sum_{\langle ij \rangle \in m}$ denotes a sum over all the nearest-neighbor pairs on the $m$-direction bond. Below we set, for a moment, $K_{xm} = K_{ym} = K_{xym}$ and $h = 0$. First, we consider the decimation which changes the length scale of 1-direction by a factor $l$ as shown in Fig. 8a. By the potential moving, we move the bond between spins to be eliminated by the decimation to the

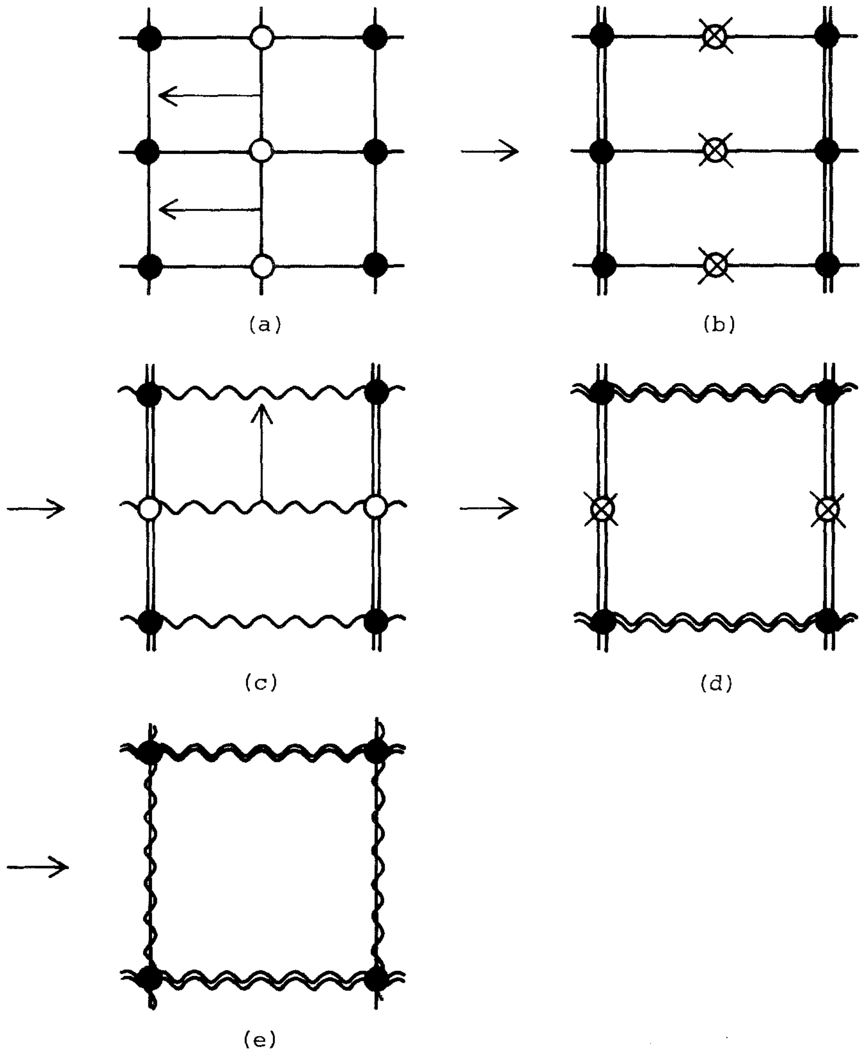



Fig. 8. The Migdal–Kadanoff transformation in two dimensions.

equivalent bonds between spins to remain after the decimation, as shown in Fig. 8a. As a result, the spins to be eliminated are on the one-dimensional chains of bonds as shown in Fig. 8b. Therefore, if $K_{xy1} = 0$, we can carry out the decimation exactly. However, we cannot carry out the decimation in the case $K_{xy1} \neq 0$. Then, using the same approximation as was used in Section 2, we carry out the decimation approximately to get a new nearest-neighbor interaction between the remaining spins. Now, we have a new lattice whose lattice constant of 1-direction is $l$ times as large as that of the original lattice; Fig. 8c shows this new lattice. The new coupling constants $\mathbf{K}'_m = \binom{K'_{zm}}{K'_{xym}}$ of $m$-direction bond are given as functions of the original coupling constants $\mathbf{K}_m = \binom{K_{zm}}{K_{xym}}$ of $m$-direction bond by $\mathbf{K}'_1 = \mathbf{R}_l(\mathbf{K}_1)$ and $\mathbf{K}'_m = l\mathbf{K}_m$, $(m = 2, 3, \ldots, d)$, where $l$ denotes the scale factor; the functions $\mathbf{R}_l(\mathbf{K}_1)$ are determined from Eq. (7). We apply this procedure which rescales the length of one direction by a factor $l$ to 2-, 3-, . . . , and $d$-directions successively as shown in Figs. 8c and 8d. Then, we have a lattice whose lattice constant of every direction is $l$ times as large as that of the original lattice. See Fig. 8e. The new coupling constants $\mathbf{K}'_m$ of $m$-direction bond are finally given by

$$\mathbf{K}'_m = l^{d-m}\mathbf{R}_l\big(l^{m-1}\mathbf{K}_m\big) \tag{A.2}$$

This is the renormalization transformation to be derived. This transformation has a problem: Each interaction of different direction has a different form of renormalization transformation. Therefore, if we start with isotropic interactions $\mathbf{K}_1 = \mathbf{K}_2 = \cdots = \mathbf{K}_d$, we get anisotropic interactions after this renormalization transformation. However, if the coupling constants $\mathbf{K}_m$ satisfy the condition

$$\mathbf{K}_1 = l\mathbf{K}_2 = \cdots = l^{m-1}\mathbf{K}_m = \cdots = l^{d-1}\mathbf{K}_d \tag{A.3}$$

then the renormalized coupling constants $\mathbf{K}'_m$ satisfy the same condition. In other words, if we assume that the coupling constants satisfy the condition (A.3), the renormalization transformation (A.2) for each direction is essentially the same and can be reduced to one of them. Therefore, in order to avoid the anisotropy in the renormalization transformation, we restrict ourselves to applying the renormalization transformation (A.2) only to a system whose coupling constants satisfy the condition (A.3).

Finally, we apply the Migdal–Kadanoff transformation to the model with a magnetic field (i.e., $h \neq 0$). There is ambiguity in moving the magnetic field term. We treat below the magnetic field term so that there may be no anisotropy in the renormalization transformation. Let

$$\mathbf{K}_m = \begin{pmatrix} K_{zm} \\ K_{xm} \\ K_{ym} \end{pmatrix}$$

denote the nearest-neighbor coupling constants of $m$-direction bond as before. We assume that the coupling constants $\mathbf{K}_m$ satisfy the condition (A.3) in order to avoid the problem of anisotropy in the case $h = 0$. In order to change the length scale of the 1-direction, we first move the bonds between spins to be eliminated to the bonds between spins to remain as shown in Fig. 9a. Then using the same approximation as in Section 2, we carry out the decimation as shown in Fig. 9b. Then the new coupling constants $\mathbf{K}'_m$ of the $m$-direction and the new magnetic field $h'$ are given by

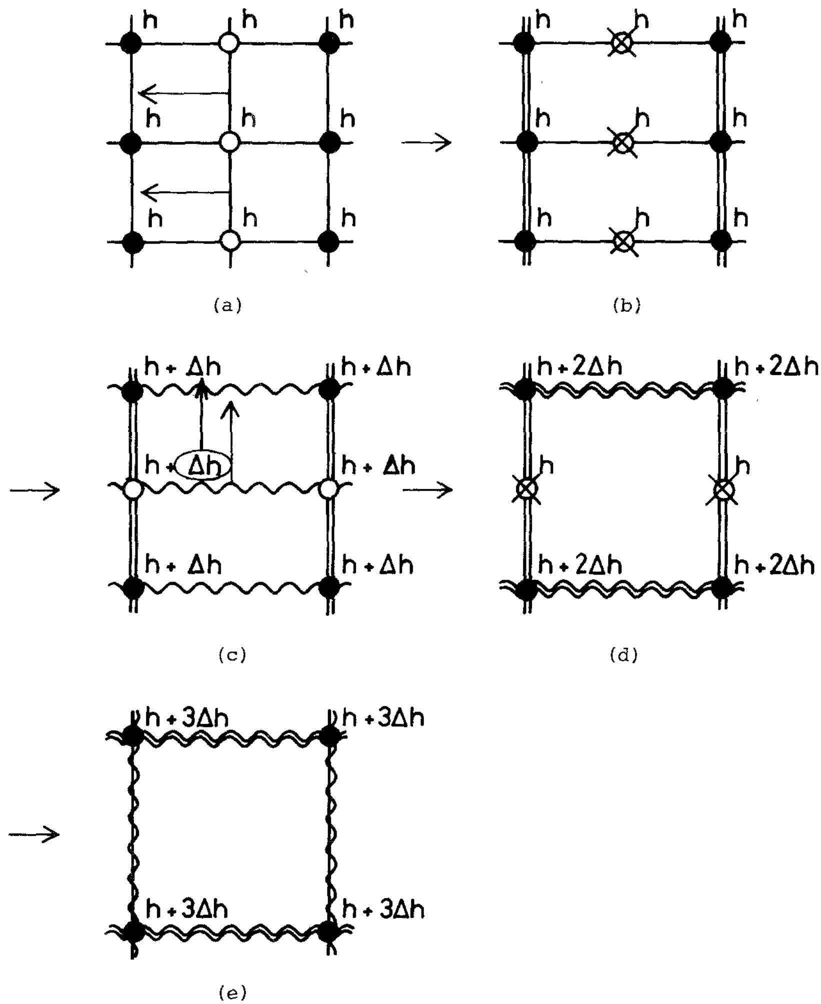



Fig. 9. The Migdal-Kadanoff transformation for the model with a magnetic field in two dimensions.

$\mathbf{K}_1' = \mathbf{R}_l(\mathbf{K}_1, h)$, $h' = r_l(\mathbf{K}_1, h)$ and $\mathbf{K}_m' = l\mathbf{K}_m$, $(m = 2, 3, \ldots, d)$, where $\mathbf{K}_1'$ and $h'$ are determined from Eq. (7) with $H(\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1) = K_{z1}\sigma_0^z\sigma_1^z + K_{x1}\sigma_0^x\sigma_1^x + K_{y1}\sigma_0^y\sigma_1^y + \frac{1}{2}h(\sigma_0^z + \sigma_1^z)$. Next, we consider changing the length scale of the 2-direction. In order to avoid the problem of anisotropy, we not only move the nearest-neighbor interactions but also the additional magnetic field $\Delta h = h' - h$ as shown in Fig. 9c. Then, we carry out the decimation approximately as shown in Fig. 9d and get the new coupling constants $\mathbf{K}_m''$ and the new magnetic field $h''$ as follows: $\mathbf{K}_2'' = R_l(\mathbf{K}_2', h) = \mathbf{K}_1'$, $h'' = r_l(\mathbf{K}_2', h) + l(h' - h) = (l + 1)h' - lh$ and $\mathbf{K}_m'' = l\mathbf{K}_m'$ $(m = 1, 3, 4, \ldots, d)$. Applying this procedure to 3-, 4-, $\ldots$, and $d$-directions successively, we obtain the final renormalized coupling constants $\mathbf{K}_m'$ of $m$-direction and magnetic field $h'$ as

$$\mathbf{K}_m' = l^{d-m}\mathbf{R}_l\left(l^{m-1}\mathbf{K}_m, h\right) \tag{A.4}$$

and

$$h' = \frac{l^d - 1}{l - 1} r_l(\mathbf{K}_1, h) - \frac{l^d - l}{l - 1} h \tag{A.5}$$

See Fig. 9e. These new coupling constants $\mathbf{K}_m'$ also satisfy the condition (A.3). Thus, we obtain the renormalization transformation without the problem of anisotropy.

## REFERENCES


1. Th. Niemeijer and J. M. J. van Leeuwen, *Phys. Rev. Lett.* **31**:1411 (1973); *Physica* **71A**:17 (1974).
2. Th. Niemeijer and J. M. J. van Leeuwen, in *Phase Transitions and Critical Phenomena*, Vol. 6, C. Domb and M. S. Green, eds. (Academic Press, London, 1976).
3. J. Rogiers and R. Dekeyser, *Phys. Rev. B* **13**:4886 (1976).
4. D. D. Betts and M. Plischke, *Can. J. Phys.* **54**:1553 (1976).
5. J. Rogiers and D. D. Betts, *Physica* **85A**:553 (1976).
6. R. Dekeyser, M. Reynaert, and M. H. Lee, *Physica* **86–88B**:627 (1977).
7. A. L. Stella and F. Toigo, *Phys. Rev. B* **17**:2343 (1978).
8. R. Dekeyser, M. Reynaert, A. L. Stella, and F. Toigo, *Phys. Rev. B* **18**:3486 (1978).
9. R. C. Brower, F. Kuttner, M. Nauenberg, and K. Subbarao, *Phys. Rev. Lett.* **38**:1231 (1977).
10. N. Honda, *Phys. Lett.* **65A**:427 (1978).
11. Z. Friedman, *Phys. Rev. Lett.* **36**:1326 (1976).
12. K. Subbarao, *Phys. Rev. Lett.* **37**:1712 (1976).
13. G. Um, *Phys. Rev. B* **15**:2736 (1977); **17**:3670 (1978).
14. R. Jullien, P. Pfeuty, J. N. Fields, and S. Doniach, *Phys. Rev. B* **18**:3568 (1978).
15. R. Jullien and P. Pfeuty, *Phys. Rev. B* **19**:4646 (1979).
16. K. A. Penson, R. Jullien, and P. Pfeuty, *Phys. Rev. B* **19**:4653 (1979).
17. K. Uzelac, P. Pfeuty, and R. Jullien, *Phys. Rev. Lett.* **43**:805 (1979).
18. R. Jullien, K. A. Penson, P. Pfeuty, and K. Uzelac, in *Ordering in Strongly Fluctuating Condensed Matter Systems*, T. Riste, ed. (Plenum Press, New York, 1980).

19. M. Suzuki and H. Takano, *Phys. Lett.* **69A**:426 (1979).
20. J. Rogiers, E. W. Grundke, and D. D. Betts, *Can. J. Phys.* **57**:1719 (1979).
21. N. D. Mermin and H. Wagner, *Phys. Rev. Lett.* **17**:1133 (1966).
22. M. Suzuki, S. Miyashita, and A. Kuroda, *Prog. Theor. Phys.* **58**:1377 (1977).
23. A. A. Migdal, *Zh. Eksp. Teor. Fiz.* **69**:1457 (1975) [*Sov. Phys. JETP* **42**:743 (1976)]; *ibid.* **69**:810 [**42**:413].
24. L. P. Kadanoff, *Ann. Phys. (N.Y.)* **100**:359 (1976).
25. M. Barma, D. Kumar, and R. B. Pandey, *J. Phys. C* **12**:L909 (1979).
26. M. Suzuki, *Commun. Math. Phys.* **51**:183 (1976).
27. R. M. Wilcox, *J. Math. Phys.* **8**:962 (1967).
28. J. C. Bonner and M. E. Fisher, *Phys. Rev.* **135**:A640 (1964).
29. S. Katsura, *Phys. Rev.* **127**:1508 (1962); *ibid.* **129**:2835 (1963).
30. K. Yamaji and J. Kondo, *J. Phys. Soc. Jpn* **35**:25 (1973).
31. G. S. Rushbrooke, G. A. Baker, Jr., and P. J. Wood, in *Phase Transitions and Critical Phenomena*, Vol. 3, C. Domb and M. S. Green, eds. (Academic Press, London, 1974).
32. T. Obokata, I. Ono, and T. Oguchi, *J. Phys. Soc. Jpn* **23**:516 (1967).
33. A. Aharony, in *Phase Transitions and Critical Phenomena*, Vol. 6, C. Domb and M. S. Green, eds. (Academic Press, London, 1976).
34. P. Pfeuty, D. Jasnow, and M. E. Fisher, *Phys. Rev. B* **10**:2088 (1974).
35. R. J. Elliott, P. Pfeuty, and C. Wood, *Phys. Rev. Lett.* **25**:443 (1970).
36. C. Domb, in *Phase Transitions and Critical Phenomena*, Vol. 3, C. Domb and M. S. Green, eds. (Academic Press, London, 1974).
37. D. D. Betts, in *Phase Transitions and Critical Phenomena*, Vol. 3, C. Domb and M. S. Green, eds. (Academic Press, London, 1974).
38. M. Suzuki, *Phys. Lett.* **38A**:23 (1972).
39. L. Onsager, *Phys. Rev.* **65**:117 (1944).
40. C. N. Yang, *Phys. Rev.* **85**:808 (1952).
41. V. L. Berezinskii, *Zh. Eksp. Teor. Fiz.* **59**:907 (1970) [*Sov. Phys. JETP* **32**:493 (1971)]; *ibid.* **61**:1144 (1971) [**34**:610 (1972)].
42. J. Zittartz, *Z. Phys.* **B23**:55, 63 (1976).
43. J. M. Kosterlitz and D. J. Thouless, *J. Phys. C* **6**:1181 (1973).
44. J. M. Kosterlitz, *J. Phys. C* **7**:1046 (1974).
45. J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, *Phys. Rev. B* **16**:1217 (1977).
46. W. J. Camp and J. P. Van Dyke, *J. Phys. C* **8**:336 (1975).
47. A. J. Guttmann, *J. Phys. A* **11**:545 (1978).
48. S. Miyashita, H. Nishimori, A. Kuroda, and M. Suzuki, *Prog. Theor. Phys.* **60**:1669 (1978).
49. S. Miyashita, *Prog. Theor. Phys.* **63**:797 (1980), and references cited therein.
50. G. Martinelli and G. Parisi, preprint (CERN, 1980).
51. T. Tatsumi, *Prog. Theor. Phys.* **65**:451 (1981).
52. H. Takano and M. Suzuki, *Physica A*, in press.